Third-order braid invariants

1991 J. Phys. A: Math. Gen. 24 4027

(http://iopscience.iop.org/0305-4470/24/17/019)

J. Phys. A: Math. Gen. 24 (1991) 4027-4036. Printed in the UK

Third-order braid invariants

M A Berger

Mathematics Department, University College London, Gower Street, London WC1E 6BT, UK

Received 30 January 1991, in final form 24 May 1991

Abstract. This report analyses the topological invariants of three-braided curves $a(t)$, $b(t)$ and $c(t)$. 3-braids are represented as a single phase curve $\tilde{\gamma}(t)$ in a two-dimensional configuration space. This configuration space consists of a set of triangular regions connected at their vertices. The curve $\tilde{\gamma}(t)$ passes through a vertex whenever $a(t)$, $b(t)$ and $c(t)$ are collinear. The sequence of vertices completely describes the braid (up to uniform twists). The length $\mathcal{T}$ of this sequence can be employed as a measure of topological complexity. The energy of a set of braided magnetic flux tubes is expected to be proportional to $\mathcal{T}^2+\mathcal{W}^2$, where $\mathcal{W}$ is the total winding number (or signed crossing number) of the braid. Second-order winding numbers are integrals of closed 1-forms like $\mathrm{d}\theta_{ab}$. This report presents a third-order winding number $\Psi(\gamma)$ which is also an integral of a closed 1-form, but which depends on relations between all three curves. The number $\Psi(\gamma)$ can be non-zero even when all the second-order winding numbers vanish. Furthermore, $\Psi(\gamma)$ bears a simple relation to the Massey third-order linking number.

1. Introduction

The braid group (Artin 1947, Birman 1974) appears in several areas of mathematics and physics, for example knot theory, statistical mechanics (e.g. Yang and Ge 1989), and the study of quantum systems with fractional statistics (Dowker 1985). Braids can also arise in classical physics—polymer chains can be braided, as can vortex lines in a turbulent fluid. Magnetic field lines in the atmosphere of a star or accretion disc are often anchored in a turbulent convection zone; a random walk of the field lines beneath the surface can braid the field lines above. These braided lines store energy which can be released during violent reconnection events (Parker 1983, Berger 1990a, b).

The differential equations governing a set of braided magnetic lines or vortex lines (or simply braided ropes) are nonlinear and three-dimensional, and hence difficult to solve. Topological invariants provide an easy way of obtaining information on the structure and energy of braided objects.

Section 2 presents a geometrical description of 3-braids in terms of a single phase curve in a configuration space, and reviews the second-order winding numbers. Section 3 employs the generators of the cohomology ring of the braid group (Arnold 1969) to derive a third-order winding number for braids. In section 4 this number is related to the Massey third-order linking number.

2. The geometry of 3-braids

We visualize a braid with three strings as follows (see figure 1): the three strings are labelled a, b and c. They stretch from the plane $t=0$ to $t=1$. Also they always move

0305-4470/91/174027+10\$03.50

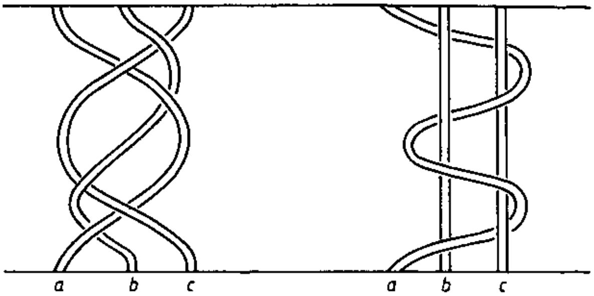

Figure 1. A braid with three strings. The two braids shown are equivalent.

upwards in t; if s is arc-length along a string, then $\mathrm{d}t/\mathrm{d}s \neq 0$. Each string can then be parametrized by t. It will be convenient to represent positions in the planes $t=$ constant by complex numbers. Thus we will let $a(t)$, $b(t)$ and $c(t)$ be complex functions of the unit interval. The braid can then be thought of as a record of the motions of three points in the complex plane.

A set of three curves $a(t)$, $b(t)$ and $c(t)$ is a *geometrical braid.* A *topological braid* is an equivalence class of geometrical braids. If one geometrical braid can be distorted into another without having strings break, and without moving the endpoints at $t=0$ and $t=1$, then they are ambient isotopic, or topologically equivalent.

We now describe the braid as a single curve

$$\gamma(t)=(a(t), b(t), c(t)) \tag{1}$$

in three complex dimensions, i.e. γ: $[0, 1] \rightarrow \mathscr{C}_3$, where

$$\mathscr{C}_3=\{(z_1, z_2, z_3)|z_i \neq z_j \text{ if } i \neq j\} \tag{2}$$

(in the notation of Birman 1974, $\mathscr{C}_3 = F_{0,3}\boldsymbol{C}$).

The configuration space $\mathscr{C}_3$ has six degrees of freedom. Much of the information carried by a curve γ in $\mathscr{C}_3$ only serves to distinguish between different geometrical braids belonging to the same topological braid. In particular, at any 'time' t, two degrees of freedom specify the centre of the triangle $\Delta abc(t)$. One further degree of freedom specifies the size of $\Delta abc(t)$. But both the position and size of $\Delta abc(t)$ can be readily varied (except at $t=0$ and $t=1$) without changing the topology.

The three remaining degrees of freedom specify the shape and orientation of the triangle $\Delta abc(t)$. These three degrees of freedom can always be specified by three winding angles $\theta_{ab}(t)$, $\theta_{bc}(t)$ and $\theta_{ca}(t)$ where, e.g.,

$$\theta_{ab}(t)=\operatorname{Im}\ln(a(t)-b(t)). \tag{3}$$

We assume $-\pi < \theta_{ij}(0) \leq \pi$; at later times the precise value of $\theta_{ij}(t)$ can be found by analytic continuation. Thus $-\infty < \theta_{ij}(t) < \infty$, and two strings winding about each other through one turn $(\theta_{ab}(1)-\theta_{ab}(0)=2\pi)$ are distinguished from two parallel strings.

Consider curves $\bar{\gamma}(t)=(\theta_{ab}, \theta_{bc}, \theta_{ca})(t)$ in a three-dimensional space with coordinates $(\theta_{ab}, \theta_{bc}, \theta_{ca})$. For the pigtail braid in figure 1, the curve $\bar{\gamma}$ forms a closed loop, since $\theta_{ab}(1)=\theta_{ab}(0)$, etc. If we deform the strings shown in figure 1, then $\bar{\gamma}$ should also deform (unless the deformation involves only uniform translations and expansions). The pigtail braid, of course, cannot be deformed into three vertical lines; this implies the impossibility of shrinking $\bar{\gamma}$ to a single point. Evidently there is some obstruction which prevents such a shrinkage. Some points are forbidden—for example no triangle can have the angles $(\theta_{ab}, \theta_{bc}, \theta_{ca}) = (0, 0, 0)$. As we will see, the $\tilde{\gamma}$ curve for figure 1 forms a loop about a forbidden region consisting of such points. But first we will reduce our configuration space by one further dimension.

The sum of the windings $\Theta(t)$ is

$$\Theta(t) = (\theta_{ab}(t) - \theta_{ab}(0)) + (\theta_{bc}(t) - \theta_{bc}(0)) + (\theta_{ca}(t) - \theta_{ca}(0)). \tag{4}$$

The total winding number

$$\mathscr{W} \equiv \Theta(1) \tag{5}$$

is a topological invariant (for Artin braids $\mathscr{W} = \pi$ times the (signed) crossing number). There is a simple correspondence between $\mathscr{W}$ and helicity integrals. Suppose each string were a vortex tube (or magnetic flux tube) with flux Φ. Then the gauge-invariant helicity of the tubes (Berger and Field 1984) would be $\mathscr{H} = \mathscr{W}\Phi^2/\pi$, plus a term arising from the twisting of vortex lines within the tubes.

Now, any given geometric braid $\gamma(t)$ can be deformed into a special form: a topologically equivalent braid which has zero winding number in the lower half, and a uniform twist in the upper half. The deformation takes two steps. First send the entire braid into the lower half, i.e. define a new braid $\gamma_1(t)$ where

$$\gamma_1(t) = \begin{cases} \gamma(2t) & t < \frac{1}{2} \\ \gamma(1) & t \geqslant \frac{1}{2}. \end{cases} \tag{6}$$

Next apply a uniform rotation to each plane t by an angle $\mu(t)$, where $\mu(0) = \mu(1) = 0$ and $\mu(t)$ is continuous $(-\infty < \mu < \infty)$. Under such a transformation $\Theta(t) \to \Theta(t) + 3\mu(t)$. For our purposes set

$$\mu(t) = \begin{cases} -\Theta(t)/3 & \text{if } 0 \leqslant t \leqslant \frac{1}{2} \\ \frac{2}{3}(t-1)\mathscr{W} & \text{if } \frac{1}{2} \leqslant t \leqslant 1. \end{cases} \tag{7}$$

The braid $\gamma_1(t)$ will then be transformed into an equivalent braid $\gamma_2(t)$ where

$$\begin{aligned} &\Theta_2(t) = 0 && \text{if } 0 \leqslant t \leqslant \tfrac{1}{2} \\ &\theta_{2ij}(t) = \theta_{2ij}(\tfrac{1}{2}) + \tfrac{2}{3}(t - \tfrac{1}{2})\mathscr{W} && \text{if } \tfrac{1}{2} \leqslant t \leqslant 1. \end{aligned} \tag{8}$$

In the lower half of the braid, the total winding is zero. In the upper half there is only a uniform twist, without any further entanglement. (In the language of group theory, uniform twists commute with all other braid elements; they compromise the centre of the braid group. The 'entangled' lower half of the braid corresponds to the braid group modulo its centre.)

By ignoring the uniform twist upper half of the braid, we reduce the number of degrees of freedom to 2. Let us examine the lower half of the braid in detail. It will be convenient to employ coordinates

$$\phi_{ij}(t) = \frac{3}{\pi}[\theta_{ij}(t) - \theta_{ij}(0)]. \tag{9}$$

Since $\Theta = 0$ for this part of the braid, equation (4) for Θ implies that the points $\tilde{\gamma}(t) = (\phi_{ab}, \phi_{bc}, \phi_{ca})(t)$ lie in the plane $\mathscr{P}_3 = \{\phi_{ab} + \phi_{bc} + \phi_{ca} = 0\}$. The curve $\gamma(t)$ for a geometrical braid in $\mathscr{C}_3$ thus determines a curve $\tilde{\gamma}(t)$ in $\mathscr{P}_3$. If it is desired to map $\gamma(t)$ to $\tilde{\gamma}(t)$ without going through the transformation in equations (6) through (8), one can simply define the variables ϕ_{ij} by

$$\phi_{ij} = 3/\pi[\theta_{ij}(t) - \theta_{ij}(0) - \tfrac{1}{3}\Theta(t)]. \tag{10}$$

Figure 2 shows the plane $\mathscr{P}_3$. The curve for figure 1 neatly makes one loop about a forbidden region. The plane $\mathscr{P}_3$ is related to the state space for windings on the thrice-punctured sphere (Lyons and McKean 1984). The forbidden regions have the shape of regular hexagons, and the allowed regions are equilateral triangles. The point at the centre of each allowed triangle corresponds to the physical triangle Δabc being equilateral.

The interior angles of a degenerate Euclidean triangle satisfy the following constraint: if one interior angle is exactly 0, then the other two interior angles must be 0 and π. The sides of the allowed regions in figure 2 correspond to angles which violate this constraint. Thus the sides themselves belong to the forbidden regions. (For example, suppose the point a is at the origin in the complex plane, the point $b = 1$, and $c = 1 + \varepsilon\lambda$ where $\varepsilon > 0$ is real and λ is complex. As $\varepsilon \to 0$ the point c approaches b. Meanwhile the interior angle at a, $\theta_{ac} - \theta_{ab} \to 0$, but the other two interior angles do not in general approach 0 or π.)

Most importantly, the vertices of the $\mathscr{P}_3$ triangles correspond to the three points $a(t)$, $b(t)$ and $c(t)$ being collinear. We can label each vertex A, B or C depending on which of the points $a(t)$, $b(t)$ or $c(t)$ is in the middle (see figure 3). The curve $\tilde{\gamma}$ passes through a sequence of vertices. This sequence can be notated as a sequence of letters, like *ABACBABC*. Repeated letters (for example *AA*) can be removed from the sequence; they correspond to $\tilde{\gamma}$ passing through a vertex and immediately coming back. A deformation of the strings can remove such a trivial path. Once repeated letters are removed, the sequence is invariant to deformations.

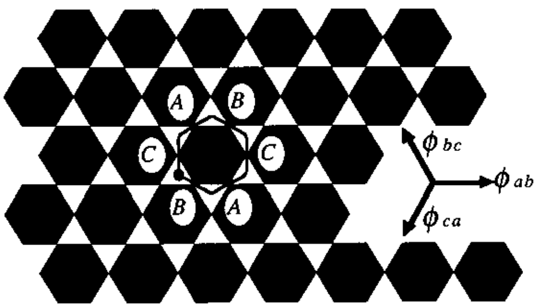

Figure 2. The plane $\mathscr{P}_3 = \{\phi_{ab} + \phi_{bc} + \phi_{ca} = 0\}$. A braid curve $\tilde{\gamma}(t)$ must stay within the triangular regions, and passes through a vertex whenever $a(t)$, $b(t)$ and $c(t)$ are collinear. The curve $\tilde{\gamma}$ for figure 1 is shown; it passes through the sequence of vertices *BACBAC*. (See also figure 4.)

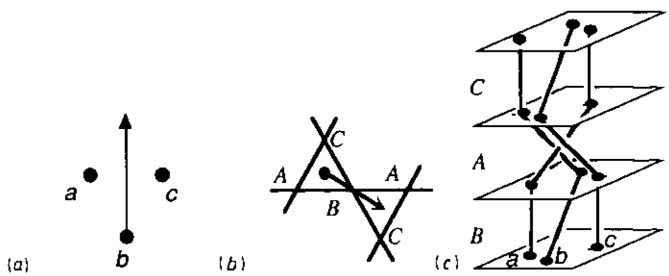

Figure 3. (a) At $t = 0$ a, b and c form an equilateral triangle with positive orientation. A single move converts Δabc to an equilateral with negative orientation. A move of type B is pictured. (b) The corresponding move in $\mathscr{P}_3$. (c) Example of a braid with sequence *BAC*.

The topology of a 3-braid can now be completely specified by giving the winding number $\mathcal{W}$ and the sequence of vertices. Let us call the length of the sequence (number of letters) the *tangling number* $\mathcal{T}$. The tangling number provides a measure of topological complexity which can be non-zero even when the winding numbers $\theta_{ij}(1)-\theta_{ij}(0)$ are zero.

Moffat (1990) points out that the minimum energy of a magnetic field whose field lines are knotted provides a measure of complexity for the knot (see also Freedman and He 1991). This idea can be extended to braided magnetic lines. Berger (1990a, b) suggests that the free energy of three braided magnetic flux tubes is proportional to $\mathcal{T}^2+\mathcal{W}^2$ (free energy means energy of braided tubes minus energy of three parallel tubes). Furthermore, if the braiding is generated by random motions in the plane, the mean square winding $\overline{\mathcal{W}^2}$ grows linearly in time, whereas $\overline{\mathcal{T}^2}$ grows quadratically.

We complete this section by suggesting a generalization of $\mathcal{T}$ to braids with n strings $z_i(t)$, $i=1,\ldots,n$. Recall that a measure of the complexity of a knot is provided by its order, which equals the minimum number of crossovers of the knot as seen in a planar projection. A projection of a braid onto a plane will also exhibit crossovers, but the number of crossovers will depend on the angle of projection. One way of eliminating this problem is to project onto a cylinder. More precisely, choose the origin in $\boldsymbol{C}$ to coincide with the centre of mass of the points at the lower plane, so that $z_1(0)+z_2(0)+\ldots+z_n(0)=0$. With little loss of generality we will also require this to be true at $t=1$. Then any braid can be distorted into a form where $z_1(t)+z_2(t)+\ldots+z_n(t)=0$ for all t. Next project the braid onto the unit cylinder, i.e. $z_i(t)\to z_i(t)/|z_i(t)|$ (if necessary deform the curves so that no points $z_i(t)=0$). The tangling number can then be defined as the minimum number of crossovers as seen in this cylindrical projection. It is simple to see that this definition yields an equivalent number to $\mathcal{T}$ for $n=3$. One advantage of a cylindrical projection is that the uniform rotation part of the braid (which is already measured by the total winding number) does not contribute to the number of crossovers.

3. Third-order winding numbers

Braids for which the three net winding numbers are equal,

$$\theta_{ab}(1)-\theta_{ab}(0)=\theta_{bc}(1)-\theta_{bc}(0)=\theta_{ca}(1)-\theta_{ca}(0) \tag{11}$$

have closed curves $\tilde{\gamma}$ in the plane $\mathcal{P}_3$. One might suspect that the number of hexagons encircled by $\tilde{\gamma}$ would be given by some simple formula. In this section we derive an integral invariant which, for closed $\tilde{\gamma}$, yields the number of encircled hexagons (actually, the number encircled counterclockwise minus the number encircled clockwise). Section 4 then considers links formed from braids where all the net winding numbers in equation (11) vanish. We show that the number of encircled hexagons is then equivalent to the third-order Massey linking number. Throughout this and the following section we will make use of differential forms (see, for example, Bott and Tu 1982). The calculations are considerably simplified by using complex-valued forms, i.e. contangent vectors to the manifold $\mathscr{C}_3$.

Arnold (1969) showed that the cohomology ring of the braid group on N curves $z_i(t)$ is generated by the 1-forms

$$\omega_{ij}=\frac{1}{2\pi\mathrm{i}}\frac{\mathrm{d}z_j-\mathrm{d}z_i}{z_j-z_i} \tag{12}$$

and their exterior products. The structure of the cohomology ring is determined by the identity

$$\omega_{ij} \wedge \omega_{jk} + \omega_{jk} \wedge \omega_{ki} + \omega_{ki} \wedge \omega_{ij} = 0 \tag{13}$$

for any triplet z_i, z_j, z_k.

For the present purposes, we set $N = 3$ and consider the curves $\gamma(t) = (a, b, c)(t)$. Let

$$\lambda_{ab}(t) = \frac{1}{2\pi \mathrm{i}} \ln \frac{b(t) - a(t)}{b(0) - a(0)} \tag{14}$$

so that $\omega_{ab} = \mathrm{d}\lambda_{ab}$ (the complex logarithm is defined as in equation (3)).

Now consider the integral along γ,

$$\Psi(\gamma) = \mathrm{Re} \int_\gamma (\lambda_{ab}\, \mathrm{d}\lambda_{bc} + \lambda_{bc}\, \mathrm{d}\lambda_{ca} + \lambda_{ca}\, \mathrm{d}\lambda_{ab}). \tag{15}$$

By equation (13) the 1-form

$$\psi = \lambda_{ab}\, \mathrm{d}\lambda_{bc} + \lambda_{bc}\, \mathrm{d}\lambda_{ca} + \lambda_{ca}\, \mathrm{d}\lambda_{ab} \tag{16}$$

is closed. As a consequence, $\Psi(\gamma)$ is the same for two homotopic curves γ_1 and γ_2 in $\mathscr{C}_3$. But any two geometric braids which belong to the same topological braid have homotopic curves. Thus $\Psi(\gamma)$ is a topological invariant, which we will call the third-order winding number.

We will now calculate $\Psi(\gamma)$. For simplicity suppose that at $t = 0$ and $t = 1$ the points a, b and c form an equilateral triangle with unit sides. We assume that at $t = 0$ the points are oriented in a positive (right-handed) sense (see figure 3(a)). First consider the upper half of the braid in equation (8), where the three points undergo a uniform twist through an angle $\mathscr{W}/3$. The curve $\gamma(t)$ moves orthogonally to $\mathscr{P}_3$, that is $\tilde{\gamma}$ stays at one point. From equation (15) the change in $\Psi(\gamma)$ is

$$\Psi(\gamma)_{\mathrm{twist}} = \frac{1}{24\pi^2} \mathscr{W}^2. \tag{17}$$

Next consider the zero twist part of the braid, corresponding to the path $\tilde{\gamma}$ in $\mathscr{P}_3$. A path through a sequence of vertices V_n, $n = 1, \ldots, \mathscr{T}$ in $\mathscr{P}_3$ can be decomposed into $\mathscr{T}$ moves, where each move begins and ends with an equilateral Δabc (as in figure 3(c)). At the middle of each move, the three points a, b, c are collinear, corresponding to $\tilde{\gamma}$ passing through a vertex. Note that the orientation of Δabc changes during each move, and recall that each vertex has type A, B or C depending on which point a, b or c is in the middle. If move n flips the orientation from positive to negative, then integrating ψ over the move gives a contribution to $\Psi(\gamma)$ of

$$f(V_n) = \tfrac{1}{18} \begin{cases} \phi_{bc}(V_n) - \phi_{ca}(V_n) & \mathrm{type}(V_n) = A \\ \phi_{ca}(V_n) - \phi_{ab}(V_n) & \mathrm{type}(V_n) = B \\ \phi_{ab}(V_n) - \phi_{bc}(V_n) & \mathrm{type}(V_n) = C. \end{cases} \tag{18}$$

For a move from negative to positive orientation, $f(V_n)$ is multiplied by -1. The total for a path γ (including the twist part) is thus

$$\Psi(\gamma) = \sum_{n=1}^{\mathscr{T}} (-1)^{n-1} f(V_n) + \frac{1}{24\pi^2} \mathscr{W}^2. \tag{19}$$

(The imaginary part of the integral of ψ has been neglected so far. However, it is simple to show that the imaginary part vanishes after any two successive moves.) The third-order winding number can be defined for any 3-braid. However, if $\tilde{\gamma}$ is a closed path in $\mathcal{P}_3$, then $\Psi(\tilde{\gamma}) = \Psi(\gamma) - \mathcal{W}^2/24\pi^2$ has a simple geometrical interpretation. The coordinates of the six vertices of the hexagon encircled by the pigtail braid of figure 1 are shown in figure 4. Equation (19) gives $\Psi(\tilde{\gamma}) = 1$ for a path encircling the hexagon counterclockwise. An arbitrary closed path can be expressed as a sum of small paths, each enclosing a single hexagon. From this one may conclude that any closed path in $\mathcal{P}_3$ yields a value for $\Psi(\tilde{\gamma})$ equal to the number of hexagons encircled counterclockwise.

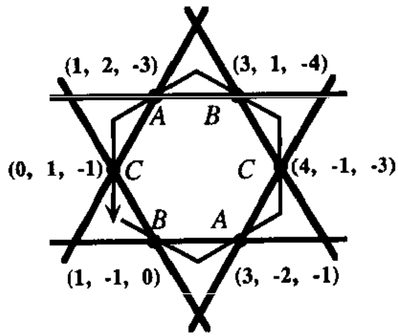

Figure 4. The coordinates $(\phi_{ab}, \phi_{bc}, \phi_{ca})$ for the braid of figure 1.

4. The Massey triple product

Links can be formed from n-braids provided that the set of points $\{z_i(0), i = 1, \ldots, n\}$ at the bottom plane is identical to the set $\{z_i(1), i = 1, \ldots, n\}$ at the top plane. The strings inside the braid are transformed into loops by identifying the top and bottom planes $t = 0$ and $t = 1$. Not all braid invariants become link invariants, however. The difficulty is that many different braids correspond to the same link. Markov's theorem (e.g. Birman 1974) describes how to move from one braid to another braid with the same associated link. This provides a powerful method of deciding whether a braid invariant is also a link invariant. Here, however, we will employ a different technique based on the Massey triple product to show that $\Psi(\gamma)$ extends to a link invariant when the three net winding numbers in equation (11) vanish.

Before proceeding, we point out that $\Psi(\gamma)$ may not be a link invariant if $\tilde{\gamma}$ is not closed. As an example, consider the braid with $\mathcal{W} = 0$ and vertex sequence $BACA$. (For simplicity we have chosen an example where $a(1) = a(0)$, $b(1) = b(0)$, $c(1) = c(0)$. With this condition each string ties to itself when top and bottom planes are identified. This allows an unambiguous assignment of the symbols a, b and c to the three strings.) Now after the planes $t = 0$ and $t = 1$ have been identified, there is no special starting point for the braid—one could start the braid sequence at any value of t. For example, the sequence $CABA$ will lead to the same link as $BACA$. But $\Psi(BACA) = 0$, whereas $\Psi(CABA) = -1$.

First we briefly review the Massey triple product, and then show how an associated topological invariant, the third-order linking number, can be calculated from $\Psi(\gamma)$.

The triple product (Massey 1958, 1968, Fenn 1983) is a mapping between cohomology classes. Using de Rham cohomology, it can be expressed as an operation on differential forms, as will be done below. Berger (1990c) provides a description in terms of three-dimensional vector calculus.

Consider three closed curves C_1, C_2, C_3 forming a link in three-dimensional space. Enclose each curve in a thin toroidal volume U_i, $i=1,2,3$. Define 1-forms A_i which satisfy $\mathrm{d}A_i=0$ outside U_i. For a curve η encircling U_i once, the integral of A_i is $\int_\eta A_i=\Phi_i$. Here Φ_i can be regarded as a magnetic flux $\mathrm{d}A_i$ contained within the volume U_i, and oriented along C_i.

Let us restrict ourselves for the moment to the region U' external to the three volumes U_i. If no two of the three curves are linked, then it can be shown that the 2-forms

$$G_1=A_2\wedge A_3 \tag{20}$$

etc, are exact; i.e. there exist F_i such that $G_i=\mathrm{d}F_i$. We can then define Massey fields

$$\begin{aligned} M_1&=A_3\wedge F_3-A_2\wedge F_2\\ M_2&=A_1\wedge F_1-A_3\wedge F_3\\ M_3&=A_2\wedge F_2-A_1\wedge F_1. \end{aligned} \tag{21}$$

Massey triple products are the equivalence classes of these fields, modulo gauge transformations of A_i and F_i.

Next define the integrals

$$m_{ij}=\int_{\partial U_i} M_j. \tag{22}$$

One can readily show that

$$m_{12}=m_{23}=m_{31}=-m_{21}=-m_{32}=-m_{13}. \tag{23}$$

A third-order linking number can then be defined by

$$\mathcal{M}=(\Phi_1\Phi_2\Phi_3)^{-1}m_{12}. \tag{24}$$

This number is invariant to gauge transformations of A_i and F_i, and is also invariant to deformations of the three curves C_i. For three unlinked curves it yields 0, while for the Borromean rings it yields ±1. Berger (1990c) showed that $\mathcal{M}$ could be computed as an integral along C_1. To do this, first drop the restriction of the fields G_i to the external region U'. Since C_i and C_j are unlinked as a pair for any two curves i and j, we can define within a neighbourhood of U_i a function $\phi_{(i)j}$ where $A_j=\mathrm{d}\phi_{(i)j}$. With the help of these functions the exact forms G_i can be defined so that they are exact everywhere. For example, in U_1

$$\begin{aligned} G_1&=A_2\wedge A_3\\ G_2&=A_3\wedge A_1+\phi_{(1)3}\,\mathrm{d}A_1\\ G_3&=A_1\wedge A_2-\phi_{(1)2}\,\mathrm{d}A_1. \end{aligned} \tag{25}$$

Next apply Stokes' theorem to equations (22) and (24) for the linking number to obtain

$$\mathcal{M}=(\Phi_1\Phi_2\Phi_3)^{-1}\int_{U_1}\mathrm{d}A_1\wedge(F_1-\phi_{(1)2}A_3) \tag{26}$$

$$=(\Phi_2\Phi_3)^{-1}\int_{C_1}(F_1-\phi_{(1)2}A_3) \tag{27}$$

$$=(\Phi_2\Phi_3)^{-1}\int_{C_1}(F_1+\phi_{(1)3}A_2). \tag{28}$$

Let us go back to braided curves. The closed curves C_i can be formed from braided curves by identifying top and bottom planes $t=0$ and $t=1$ (the braid of figure 1 becomes the Borromean rings). If the second-order winding numbers vanish, $\theta_{ij}(1)-\theta_{ij}(0)=0$, then the linkage between any two closed curves will also vanish. Thus a third-order linking number exists. We now wish to show that equation (28) for $\mathcal{M}$ is equivalent to equation (15) for $\Psi(\gamma)$.

We first comb the braid. Artin (1947) show that any braid can be put into a unique form. For 3-braids this form consists of two parts: in one half of the braid, curves $b(t)$ and $c(t)$ are vertical (the points b and c are fixed), while curve $a(t)$ winds about b and c. In the second half of the braid a is fixed, while $b(t)$ twists about c, but not about a. For our purposes the second half of the braid is not present; otherwise $\theta_{bc}(1)-\theta_{bc}(0)\neq 0$. The result is that any 3-braid with vanishing second-order winding numbers can be distorted into a form where b and c are fixed.

Next make the correspondence with equation (28) for $\mathcal{M}$, using subscripts a, b, c rather than 1, 2, 3. We need to find F_a, $\phi_{(a)c}$ and A_b. First, because b is vertical, the 1-form A_b at the point $a(t)$ can be chosen to be

$$A_b(a(t))=\frac{\Phi_b}{2\pi}\,\mathrm{d}\theta_{ab}(a(t)) \tag{29}$$

(and similarly for A_c). We can then let $\phi_{(a)c}=\Phi_c\theta_{ca}/2\pi$. Furthermore

$$G_a=\mathrm{d}F_a=A_b\wedge A_c=\frac{\Phi_b\Phi_c}{4\pi^2}\,\mathrm{d}\theta_{ab}\wedge\mathrm{d}\theta_{ca}. \tag{30}$$

Now,

$$\mathrm{d}\theta_{ab}\wedge\mathrm{d}\theta_{ca}=\frac{1}{r_{ab}r_{ca}}\,\mathrm{d}r_{ab}\wedge\mathrm{d}r_{ca} \tag{31}$$

so a suitable choice for F_a is

$$F_a=-\frac{\Phi_b\Phi_c}{4\pi^2}\ln r_{ca}\,\mathrm{d}\ln r_{ab}. \tag{32}$$

With these expressions for F_a, $\phi_{(a)c}$ and A_b, equation (28) gives

$$\mathcal{M}=\frac{1}{4\pi^2}\int_a(-\ln r_{ca}\,\mathrm{d}\ln r_{ab}+\theta_{ca}\,\mathrm{d}\theta_{ab}). \tag{33}$$

The third-order winding number $\Psi(\gamma)$ for fixed b and c (equations 14 and 15) is

$$\Psi(\gamma)=\mathrm{Re}\int_a\lambda_{ca}\,\mathrm{d}\lambda_{ab} \tag{34}$$

$$=-\frac{1}{4\pi^2}\int_a((\ln r_{ca}(t)-\ln r_{ca}(0))\,\mathrm{d}\ln r_{ab}+(\theta_{ca}(t)-\theta_{ca}(0))\,\mathrm{d}\theta_{ab}). \tag{35}$$

The terms involving $\ln r_{ca}(0)$ and $\theta_{ca}(0)$ vanish after integration. Comparing equations (33) and (35) gives the final result, $\mathcal{M}=\Psi(\gamma)$.

Acknowledgment

I greatly appreciate discussions with R Fenn, F Johnson, T Lyons and A Wright.

References

Arnold V I 1969 *Math. Notes Acad. Sci. USSR* **5** 227–32 (Engl. transl. 1974 *Math. Notes* **5** 138)

Artin E 1947 *Ann. Math.* **2** 101–26

Berger M A 1990a Third-order invariants of randomly braided curves *Topological Fluid Mechanics* ed H K Moffatt and A Tsinober (Cambridge: Cambridge University Press) pp 440–8

—— 1990b Braided magnetic flux ropes and coronal heating *Physics of Magnetic Flux Ropes* ed C Russell, E Priest and L C Lee (American Geophysical Union) pp 251–6

—— 1990c *J. Phys. A: Math. Gen.* **23** 2787–93

Berger M A and Field G B 1984 *J. Fluid Mech.* **147** 133–48

Birman J S 1974 *Braids, Links and Mapping Class Groups* (Princeton, NJ: Princeton University Press)

Bott R and Tu L W 1982 *Differential Forms in Algebraic Topology* (New York: Springer)

Dowker J S 1985 *J. Phys. A: Math. Gen.* **18** 3521

Fenn R A 1983 Techniques of geometric topology *London Math. Soc. Lecture Note Series* vol **57** (Cambridge: Cambridge University Press)

Freedman M and He Z X 1991 *Ann. Math.* in press

Lyons T J and McKean H P 1984 *Adv. Math.* **51** 212–25

Massey W S 1958 Some higher order cohomology operations *Symp. Int. Topologia Algebraica, Mexico (UNESCO)*

Massey W S 1968 Higher order linking numbers *Proc. Conf. on Algebraic Topology, University of Illinois at Chicago Circle* pp 174–205

Moffatt H K 1990 *Nature* **347** 367–9

Parker E N 1983 *Astrophys. J.* **264** 642

Yang C N and Ge M L 1989 *Braid Group, Knot Theory and Statistical Mechanics* (Singapore: World Scientific)